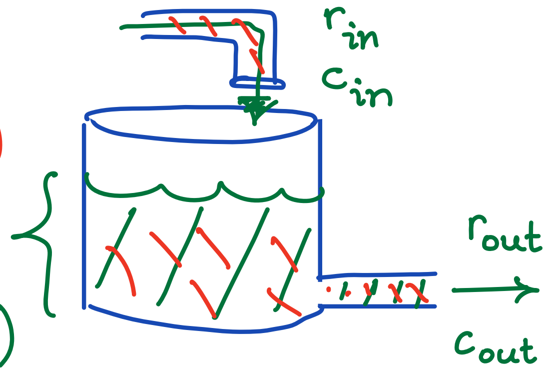
Sec 1.5(2) Mixture Problems

Setup:



$A(t) =$ amt of chem. (lbs, kg...)

$V(t) =$ volume of sol. (gal, L, ...)

r_{in}, r_{out} = fluid flow rate (gal/min, L/hr ...)

c_{in}, c_{out} = concentration $\left(\frac{lbs}{gal}, \ldots\right)$

During time Δt, $\Delta A = (A_{in} - A_{out})$

$$= \underbrace{(r_{in} \cdot c_{in} - r_{out} \cdot c_{out})}_{lbs/min,\ etc\ldots} \Delta t$$

$$\frac{\Delta A}{\Delta t} = r_{in} c_{in} - r_{out} c_{out}$$

If sol. is "uniform", we can assume $c_{out} = A(t)/V(t)$

Take $\lim\limits_{\Delta t \to 0} \Rightarrow$

$$\boxed{\begin{aligned} \frac{dA}{dt} &= r_{in} c_{in} - \frac{r_{out}}{V(t)} A(t) \\ \text{aka} \quad \frac{dA}{dt} + \frac{r_{out}}{V(t)} A(t) &= r_{in} c_{in} \end{aligned}}$$

$(A' + P(t)A = Q(t))$

Ex A tank initially contains 40 lbs salt in 400 gal of water.

Water with $\frac{1}{2}$ lb/gal comes in @ 4 $\frac{gal}{min}$,

uniform sol. goes out @ 4 $\frac{gal}{min}$.

Model amt of salt in solution over time $(A(t))$

$A(t) =$ lbs

$$\frac{dA}{dt} = r_{in}\, c_{in} - r_{out}\, c_{out} \rightarrow \text{"uniform"}$$

$$= (4)(1/2) - (4)\cdot \frac{A(t)}{V(t)}$$

$$\bigstar \quad \underline{V(t) = V(0) + (r_{in} - r_{out})t}$$

$$= 400 + (4-4)t$$

$$= 400$$

$$\frac{dA}{dt} + \frac{4}{400} A(t) = 2$$

$$A' + P(t)A(t) = Q(t)$$

$$P(t) = \frac{1}{100}, \quad Q(t) = 2$$

$$\rho(t) = e^{\int P(t)\,dt} = e^{\int \frac{1}{100}\,dt} = e^{t/100}$$

$$\rho \cdot A' + \rho P A = \rho Q$$

$$\rho A' + \rho' A = \rho Q$$

$$\int \frac{d}{dt}\left[e^{t/100} A\right] dt = \int e^{t/100} \cdot 2\,dt$$

$$\Rightarrow \quad e^{t/100} A = 200\, e^{\frac{t}{100}} + C$$

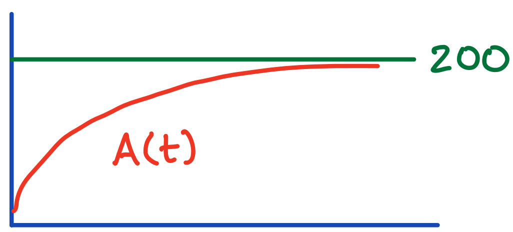
$$\Rightarrow \quad \underline{A = 200 + C e^{-t/100}} \quad \text{(gen sol.)}$$

Need C: know $A(0) = 40$ lbs

$$40 = A(0) = 200 + Ce^{0}$$

$$C = -160$$

so $\underline{A(t) = 200 - 160\, e^{-x/100}}$



IVP was $\left\{ \begin{array}{l} A' + \frac{1}{100} A = 2 \\ A(0) = 40 \end{array} \right\}$

Ex What if now the r_{out} is 6 gal/min?

$$\frac{dA}{dt} = r_{in} c_{in} - r_{out} \cdot \frac{A(t)}{V(t)}$$

(tank empty @ $t=200$
so assume $t \leq 200$)

$V(t) = V_0 + (4-6)t = 400 - 2t$ so

$$\left\{ \frac{dA}{dt} + \frac{6}{400-2t} A(t) = 2, \quad A(0) = 40 \right\}$$

$P(t) = \frac{3}{200-t} \quad Q(t) = 2$

$$\rho(t) = e^{\int P(t)\,dt} = e^{\int \frac{3}{200-t}\,dt} = e^{-3\ln(200-t)}$$
$$= (200-t)^{-3}$$

$$\rho \cdot A' + \rho P A = \rho \cdot Q$$
$$\int \frac{d}{dt}\left[\rho \cdot A\right] dt = \int \frac{2}{(200-t)^3}\,dt$$

$$(200-t)^{-3} A = (200-t)^{-2} + C$$

$$\underline{A = (200-t)^1 + C(200-t)^3} \quad \text{(gen)}$$

$40 = A(0) = 200 + C \cdot (200)^3 \quad \cdots \quad C = \frac{-160}{(200)^3}$

So $\boxed{A(t) = 200 - t - \frac{160}{200^3}(200-t)^3}$

Max $A(t)$? (take $A'(t) = 0$ ...)